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# Algebraicity of cycles on smooth manifolds

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**Abstract** According to the Nash–Tognoli theorem, each compact smooth manifold $M$ is diffeomorphic to a nonsingular real algebraic set, called an algebraic model of $M$. It is interesting to investigate to what extent algebraic and differential topology of compact smooth manifolds can be transferred into the algebraic-geometric setting. Many results, examples and counterexamples depend on the detailed study of the homology classes represented by algebraic subsets of $X$, as $X$ runs through the class of all algebraic models of $M$. The present paper contains several new results concerning such algebraic homology classes. In particular, a complete solution in codimension 2 and strong results in codimensions 3 and 4.

**Keywords** Real algebraic set · Algebraic homology class · Algebraic model of a smooth manifold

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## 1 Introduction and main results

There is a large research program whose goal is to transfer, as far as possible, algebraic and differential topology of compact smooth (of class $\mathcal{C}^\infty$) manifolds into the algebraic-geometric setting. The origins of this program go back to 1952 and the celebrated paper of J. Nash on real algebraic manifolds [53] (cf. also [16, Theorem 14.1.8]). Nash's result and conjectures inspired several mathematicians, but despite their efforts, no significant progress was made for 20 years (cf. [34] for historical

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remarks). A breakthrough came in 1973 due to Tognoli [63], who proved one of Nash's conjectures (cf. also [16, Theorem 14.1.10]). According to Tognoli's theorem, every compact smooth manifold $M$ is diffeomorphic to a nonsingular real algebraic set (in $\mathbb{R}^n$ for some $n$), called an *algebraic model* of $M$. A projective version of this theorem was proved in 1976 by King [35]. Actually, both [63] and [35] contain much stronger results, concerning approximation of smooth manifolds by algebraic sets, as suggested in [53]. Remarkable refinements of [35,63] can be found in the contributions from the 1980s and 1990s of two pairs of researchers, Akbulut–King [1–3,6–8] and Benedetti–Tognoli [13,14]. If some topological objects such as smooth submanifolds, vector bundles, homology or cohomology classes are attached to $M$, it is interesting to investigate whether or not there exists an algebraic model of $M$ on which the corresponding objects admit an algebraic description. Important positive results are known for smooth submanifolds [2,13] and vector bundles [13,14]. Contrary to initial expectations, expressed explicitly in [2,3], the situation is drastically different for homology and cohomology classes, where obstructions appear [12,20,41,42,61]. This on the one hand imposes limitations and on the other hand leads to challenging problems considered below.

Let $X$ be a compact nonsingular real algebraic set. A homology class in $H_d(X;\mathbb{Z}/2)$ is said to be *algebraic* if it can be represented by a $d$-dimensional algebraic subset of $X$ (cf. [27] and [8,16,22]). The set $H_d^{\mathrm{alg}}(X;\mathbb{Z}/2)$ of all algebraic homology classes in $H_d(X;\mathbb{Z}/2)$ forms a subgroup. Early papers dealing with algebraic homology classes provided examples of $X$ with $H_d^{\mathrm{alg}}(X;\mathbb{Z}/2) \neq H_d(X;\mathbb{Z}/2)$ for some $d$, $1 \leq d \leq \dim X - 1$ (cf. [3,14,15,36,54,57]). For technical reasons, it is often preferable to work with cohomology rather than homology. The subgroup $H^k_{\mathrm{alg}}(X;\mathbb{Z}/2)$ of *algebraic cohomology classes* in $H^k(X;\mathbb{Z}/2)$ is by definition the inverse image of $H_{n-k}^{\mathrm{alg}}(X;\mathbb{Z}/2)$ under the Poincaré duality isomorphism $H^k(X;\mathbb{Z}/2) \to H_{n-k}(X;\mathbb{Z}/2)$, where $n = \dim X$. In particular, $H^n_{\mathrm{alg}}(X;\mathbb{Z}/2) = H^n(X;\mathbb{Z}/2)$. The direct sum

$$H^*_{\mathrm{alg}}(X;\mathbb{Z}/2) = \bigoplus_{k\geq 0} H^k_{\mathrm{alg}}(X;\mathbb{Z}/2)$$

is a subring of the cohomology ring $H^*(X;\mathbb{Z}/2)$, containing the Stiefel–Whitney classes $w_k(X)$ of $X$ for $k \geq 0$ (cf. [27] and, for purely topological proofs, [4,15,56]). Consequently, $H^*_{\mathrm{alg}}(X;\mathbb{Z}/2)$ contains the subring of $H^*(X;\mathbb{Z}/2)$ generated by $H^n(X;\mathbb{Z}/2)$ and $w_k(X)$ for $k \geq 0$. What other, if any, cohomology classes belong to $H^*_{\mathrm{alg}}(X;\mathbb{Z}/2)$ depends in a very subtle way on the algebraic-geometric properties of $X$ (cf. [21,31,49–51,58,65,66]). The groups $H_d^{\mathrm{alg}}(-;\mathbb{Z}/2)$ and $H^k_{\mathrm{alg}}(-;\mathbb{Z}/2)$ are closely related via the cycle maps to the Chow groups of quasiprojective schemes over $\mathbb{R}$ and to the equivariant cohomology of the set of complex points of such schemes (cf. [27,28,33,36,48,66]). They play a crucial role in the research program described at the beginning (cf. [1,3–5,8–18,20,23–26,37–48,55,56,61,64] and, for a short survey, [22]).

Numerous results, examples and counterexamples in the papers just cited required information on algebraic homology and cohomology classes on various algebraic models of a given compact smooth manifold $M$. According to [19], $M$ has an uncountable family of pairwise nonisomorphic algebraic models whenever $\dim M \geq 1$. However, $M$ may not admit any algebraic model $X$ with $H^*_{\mathrm{alg}}(X; \mathbb{Z}/2) = H^*(X; \mathbb{Z}/2)$ (see the remarks preceding Corollary 1.3). In order to avoid awkward repetitions, if $X$ is an algebraic model of $M$ and $\varphi\colon X \to M$ is a smooth diffeomorphism, the pair $(X, \varphi)$ *will also be called an algebraic model of* $M$. A subring $A$ of $H^*(M; \mathbb{Z}/2)$ (only subrings containing the identity element $1 \in H^0(M; \mathbb{Z}/2)$ are considered) is said to be *algebraically realizable* if there exists an algebraic model $(X, \varphi)$ of $M$ with $\varphi^*(A) \subseteq H^*_{\mathrm{alg}}(X; \mathbb{Z}/2)$. An important algebraically realizable subring of $H^*(M; \mathbb{Z}/2)$ is identified in [13, Theorem 4, Remark 8]. It is the subring $A(M)$ generated by the Stiefel–Whitney classes of all real vector bundles on $M$ and the cohomology classes corresponding via the Poincaré duality to the homology classes represented by all compact smooth submanifolds of $M$. A conjecture posed in [12] asserts that every algebraically realizable subring of $H^*(M; \mathbb{Z}/2)$ is contained in $A(M)$. The conjecture is true if $\dim M \leq 7$, but in higher dimensions, it is not even known whether or not there is a largest algebraically realizable subring of $H^*(M; \mathbb{Z}/2)$ (cf. [41] for comments and conjectures).

The finer problem that of finding a characterization of the subrings $A$ of $H^*(M; \mathbb{Z}/2)$ for which there exists an algebraic model $(X, \varphi)$ of $M$ with $\varphi^*(A) = H^*_{\mathrm{alg}}(X; \mathbb{Z}/2)$ is wide open if $\dim M \geq 3$ (it is trivial if $\dim M \leq 1$, while its solution readily follows from [42, Corollary 1.12] for $M$ connected of dimension 2). The problem is unsolved even for $A$ contained in $A(M)$, when there are no obstructions to algebraic realizability of $A$. This paper provides partial solutions for a large class of subrings of $A(M)$ (cf. Theorems 1.1, 1.7, 2.10 and Corollaries 1.2, 1.4, 2.5, 1.8, 1.10).

The analogous problem of finding, for a fixed positive integer $r$, a characterization of the subgroups $G$ of $H^r(M; \mathbb{Z}/2)$ for which there exists an algebraic model $(X, \varphi)$ of $M$ with $\varphi^*(G) = H^r_{\mathrm{alg}}(X; \mathbb{Z}/2)$ is more tractable. It is completely solved in [17, Theorems 1.2 and 1.3] and [42, Corollary 1.12] for $r = 1$. The present paper contains a complete solution, under the assumption $\dim M \geq 5$, for $r = 2$ (cf. Corollary 1.3) and several partial results for $r \geq 3$ (cf. Corollaries 1.6, 1.9 and 1.11). A necessary condition for the existence of such model $(X, \varphi)$ is that all cup products $w_{i_1}(M) \cup \cdots \cup w_{i_p}(M)$ be in $G$, where $i_1, \ldots, i_p$ are nonnegative integers with $i_1 + \cdots + i_p = r$.

As the initial step, a suitable class of subrings of $H^*(M; \mathbb{Z}/2)$ will be defined.

If $h\colon M \to P$ is a smooth map into a compact smooth manifold $P$, then a standard transversality argument implies that $h^*(A(P)) \subseteq A(M)$ (cf. also [27, Proposition 2.15]). A subring $B$ of $H^*(M; \mathbb{Z}/2)$ is said to be *full* if $B = h^*(H^*(P; \mathbb{Z}/2))$ for some $h\colon M \to P$ with $A(P) = H^*(P; \mathbb{Z}/2)$. Every full subring is contained in $A(M)$.

For any collection $F$ of real vector bundles on $M$, the subring $F(M)$ generated by the Stiefel–Whitney classes of the vector bundles in $F$ is a full subring of $H^*(M; \mathbb{Z}/2)$. Indeed, the collection $F$ can be assumed to be finite, the set $H^*(M; \mathbb{Z}/2)$ being finite, and hence, the assertion readily follows by making use of smooth classifying maps and Künneth's theorem (cf. [30,32,59]).

For any subring $B$ and any subset $T$ of $H^*(M; \mathbb{Z}/2)$, let $B[T]$ denote the extension of $B$ by $T$, that is, the subring of $H^*(M; \mathbb{Z}/2)$ generated by $B$ and $T$. A cohomology class in $H^*(M; \mathbb{Z}/2)$ will be called *regular* if it corresponds via the Poincaré duality to a homology class in $H_*(M; \mathbb{Z}/2)$ represented by a compact smooth submanifold of $M$. The subset $T$ will be called *regular* if each cohomology class in $T$ is regular. A subring of $H^*(M; \mathbb{Z}/2)$ that is the extension of a full subring by a regular subset is said to be *admissible*. An admissible subring $A$ is said to be $r$-*admissible*, where $r$ is a nonnegative integer, if it can be written as $A = B[T]$ for some full subring $B$ and some regular subset $T$, with $T$ disjoint from $H^i(M; \mathbb{Z}/2)$ for $0 \leq i \leq r-1$. Thus, admissible is the same as 0-admissible. By a transversality argument, each admissible subring $A$ can be written as $A = F(M)[T]$, where $F$ is a finite collection of real vector bundles and $T$ is a regular subset. In particular, the definitions of an admissible subring used here and in [45] are equivalent. The largest admissible subring is $A(M)$. If $\dim M \leq 5$, then each cohomology class in $H^*(M; \mathbb{Z}/2)$ is regular [62, Théorème II.26], and hence, every subring of $H^*(M; \mathbb{Z}/2)$ is admissible.

Relationships between admissible subrings and $H^*_{\mathrm{alg}}(-; \mathbb{Z}/2)$ are investigated below. The main results, whose proofs are postponed until Sect. 2, are Theorems 1.1 and 1.7. Their significance is elaborated upon in a series of corollaries. Some simple topological facts, contained in Proposition 1.12, are also required for the derivation of the corollaries.

As usual, the $i$th Steenrod square operation will be denoted by $\mathrm{Sq}^i$. Only $\mathrm{Sq}^1$ is used in Sect. 1.

For any nonnegative integer $k$ and any subring $A$ of $H^*(M; \mathbb{Z}/2)$, let

$$A^k := A \cap H^k(M; \mathbb{Z}/2).$$

**Theorem 1.1** *Let $M$ be a compact connected smooth manifold and let $r$ be a positive integer with $2r+1 \leq \dim M$. For an $r$-admissible subring $A$ of $H^*(M; \mathbb{Z}/2)$ with $A^i = 0$ for $1 \leq i \leq r-2$, the following conditions are equivalent:*

(a) *There exists an algebraic model $(X, \varphi)$ of $M$ satisfying*

$$\varphi^*(A) \subseteq H^*(X; \mathbb{Z}/2) \quad \text{and} \quad \varphi^*(A^k) = H^k_{\mathrm{alg}}(X; \mathbb{Z}/2) \quad \text{for} \quad 0 \leq k \leq r.$$

(b) *$w_k(M)$ is in $A^k$ for $0 \leq k \leq r$.*

Of course, the condition $A^i = 0$ for $1 \leq i \leq r-2$ is vacuous if $r = 1$ or $r = 2$. If $r = 1$, then Theorem 1.1 is a minor improvement upon [17, Theorems 1.2 and 1.3]. The case $r = 2$ is much more interesting.

**Corollary 1.2** *Let $M$ be a compact connected smooth manifold of dimension at least 5. For an admissible subring $A$ of $H^*(M; \mathbb{Z}/2)$, the following conditions are equivalent:*

(a) *There exists an algebraic model $(X, \varphi)$ of $M$ satisfying*

$$\varphi^*(A) \subseteq H^*_{\mathrm{alg}}(X; \mathbb{Z}/2) \quad \text{and} \quad \varphi^*(A^k) = H^k_{\mathrm{alg}}(X; \mathbb{Z}/2) \quad \text{for } k = 0, 1, 2.$$

(b) *$w_k(M)$ is in $A^k$ for $k = 0, 1, 2$.*

*Proof* According to Proposition 1.12($\mathrm{p}_1$), each admissible subring is 2-admissible, and hence, it suffices to apply Theorem 1.1 with $r = 2$. □

Corollary 1.2 was proved in [45] for $M$ with homology group $H_{m-2}(M;\mathbb{Z})$ having no 2-torsion, where $m = \dim M$. This additional assumption removed the main difficulty in the proof.

It is interesting to extract from Corollary 1.2 and previously known results information on the behavior of $H^2_{\mathrm{alg}}(-;\mathbb{Z}/2)$. Let $A^r(M) := A(M)^r$. According to [20,61], for any compact smooth manifold $M$, the group $A^2(M)$ can be described as follows: $A^2(M) = W^2(M)$, where

$$W^2(M) := \{v \in H^2(M;\mathbb{Z}/2) \mid v = w_2(\xi) \text{ for some real vector bundle } \xi \text{ on } M \text{ with } w_1(\xi) = 0\}$$

and $w_k(\xi)$ denotes the $k$th Stiefel–Whitney class of $\xi$ for $k \geq 0$. Thus, $W^2(M) = H^2(M;\mathbb{Z}/2)$ if $\dim M \leq 5$. However, for each integer $n \geq 6$, there exists an $n$-dimensional compact connected smooth manifold $N$ with $W^2(N) \neq H^2(N,\mathbb{Z}/2)$ [61]. On the other hand,

$$H^2_{\mathrm{alg}}(X;\mathbb{Z}/2) \subseteq W^2(X)$$

for every compact nonsingular real algebraic set $X$ (cf. [12,18] and, for an elementary topological proof, [23]). In particular, $H^2_{\mathrm{alg}}(Y;\mathbb{Z}/2) \neq H^2(Y;\mathbb{Z}/2)$ for every algebraic model $Y$ of $N$.

**Corollary 1.3** *Let $M$ be a compact connected smooth manifold of dimension at least* 5. *For a subgroup $G$ of $H^2(M;\mathbb{Z}/2)$, the following conditions are equivalent:*

(a) *There exists an algebraic model $(X, \varphi)$ of $M$ satisfying*

$$\varphi^*(G) = H^2_{\mathrm{alg}}(X;\mathbb{Z}/2).$$

(b) *$w_1(M) \cup w_1(M)$ and $w_2(M)$ are in $G$, and $G \subseteq W^2(M)$.*

*Proof* If (a) holds, then $w_1(X) \cup w_1(X)$ and $w_2(X)$ belong to $\varphi^*(G)$, and $\varphi^*(G) \subseteq W^2(X)$. Hence, (b) follows.

Suppose that (b) holds. For each cohomology class $v$ in $W^2(M)$, let $\xi_v$ be a real vector bundle on $M$ with $w_1(\xi_v) = 0$ and $w_2(\xi_v) = v$. Let $F$ be the collection consisting of the tangent bundle to $M$ and the $\xi_v$ for $v$ in $G$. The subring $A := F(M)$ of $H^*(M,\mathbb{Z}/2)$ is admissible, $A^2 = G$, and $w_i(M)$ is in $A^i$ for $i \geq 0$. Hence, Corollary 1.2 implies that (a) is satisfied. □

Corollary 1.3 was already conjectured in [20], but proved there only for $M$ orientable, that is, $w_1(M) = 0$. In [40], Corollary 1.3 was proved under very restrictive assumptions on the group $H^{m-2}(M;\mathbb{Z}/2)$, where $m = \dim M$. The methods used in [20,40] do not work without these extra hypotheses.

There is also a version of Corollary 1.2 for an arbitrary, not necessarily admissible, subring.

**Corollary 1.4** *Let $M$ be a compact connected smooth manifold of dimension at least* 5. *For a subring $A$ of $H^*(M;\mathbb{Z}/2)$, the following conditions are equivalent:*

(a) *There exists an algebraic model* $(X, \varphi)$ *of* $M$ *satisfying*

$$\varphi^*(A^k) = H^k_{\mathrm{alg}}(X; \mathbb{Z}/2) \quad \textit{for } k = 0, 1, 2.$$

(b) $w_k(M)$ *is in* $A^k$ *for* $k = 0, 1, 2$, *and* $A^2 \subseteq W^2(M)$.

*Proof* It is already explained that (a) implies (b).

Suppose now that (b) holds. Each cohomology class $u$ in $H^1(M; \mathbb{Z}/2)$ can be written as $u = w_1(\gamma_u)$ for some real line bundle $\gamma_u$ on $M$. Similarly, each cohomology class $v$ in $W^2(M)$ can be written as $v = w_2(\xi_v)$ for some real vector bundle $\xi_v$ on $M$ with $w_1(\xi_v) = 0$. Let $F$ be the collection consisting of $\gamma_u$ for $u$ in $A^1$, $\xi_v$ for $v$ in $A^2$ and the tangent bundle to $M$. The subring $C := F(M)$ of $H^*(M; \mathbb{Z}/2)$ is admissible with $C^k = A^k$ for $k = 0, 1, 2$. Corollary 1.2 applied to the subring $C$ implies (a). □

Theorem 1.1 with $r = 3$ implies the following:

**Corollary 1.5** *Let* $M$ *be a compact connected orientable smooth manifold of dimension at least* 7. *For an admissible subring* $A$ *of* $H^*(M; \mathbb{Z}/2)$ *with* $A^1 = 0$, *the following conditions are equivalent:*

(a) *There exists an algebraic model* $(X, \varphi)$ *of* $M$ *satisfying*

$$\varphi^*(A) \subseteq H^*_{\mathrm{alg}}(X; \mathbb{Z}/2) \quad \textit{and} \quad \varphi^*(A^k) = H^k_{\mathrm{alg}}(X; \mathbb{Z}/2) \quad \textit{for } k = 0, 1, 2, 3.$$

(b) $w_i(M)$ *is in* $A^i$ *for* $i = 2, 3$, *and* $\mathrm{Sq}^1(A^2) \subseteq A^3$.

*Proof* According to [4, Theorem 6.6], $\mathrm{Sq}^1(H^2_{\mathrm{alg}}(-; \mathbb{Z}/2)) \subseteq H^3_{\mathrm{alg}}(-; \mathbb{Z}/2)$, and therefore, (a) implies (b).

Suppose now that (b) holds. By Proposition 1.12($\mathrm{p}_2$), there exists a 3-admissible subring $\overline{A}$ of $H^*(M; \mathbb{Z}/2)$ such that $A \subseteq \overline{A}$ and $A^k = \overline{A}^k$ for $k = 0, 1, 2, 3$. The orientability of $M$ implies $w_1(M) = 0$. Hence, (a) follows by applying Theorem 1.1 with $r = 3$ to the subring $\overline{A}$. □

It would be interesting, but very hard, to extend Corollary 1.3 to subgroups of $H^r(M; \mathbb{Z}/2)$ with $r \geq 3$. The following partial result is available.

**Corollary 1.6** *Let* $M$ *be a compact connected smooth manifold and let* $r \geq 3$ *be an integer with* $2r + 1 \leq \dim M$. *Assume that* $w_i(M) = 0$ *for* $1 \leq i \leq r - 2$. *If* $G$ *is a subgroup of* $H^r(M; \mathbb{Z}/2)$ *generated by some regular cohomology classes and* $w_r(M)$, *then there exists an algebraic model* $(X, \varphi)$ *of* $M$ *satisfying* $\varphi^*(G) = H^r_{\mathrm{alg}}(X; \mathbb{Z}/2)$.

*Proof* The subring $A$ of $H^*(M; \mathbb{Z}/2)$ generated by $G$ and the cohomology classes $w_k(M)$ for $k \geq 0$ is $r$-admissible. Moreover, $A^i = 0$ for $1 \leq i \leq r - 2$ and $A^r = G$. It remains to apply Theorem 1.1. □

If $r = 3$ in Corollary 1.6, then the condition $w_i(M) = 0$ for $1 \leq i \leq r - 2$ is equivalent to the orientability of $M$.

**Theorem 1.7** *Let $M$ be a compact connected smooth manifold whose homology group $H_{r-1}(M;\mathbb{Z})$ has no 2-torsion for some integer $r \geq 3$ with $2r+1 \leq \dim M$. Let $A$ be an $r$-admissible subring of $H^*(M;\mathbb{Z}/2)$ with $A^i = 0$ for $1 \leq i \leq r-4$. Assume that $w_j(M)$ is in $A^j$ for $0 \leq j \leq r$. Then, there exists an algebraic model $(X,\varphi)$ of $M$ satisfying*

$$\varphi^*(A) \subseteq H^*_{\mathrm{alg}}(X;\mathbb{Z}/2) \quad \textit{and}$$
$$\varphi^*(A^k) = H^k_{\mathrm{alg}}(X;\mathbb{Z}/2) \quad \textit{for } k \in \{0,1,\dots,r-2,r\} \cup \{2\}.$$

*Moreover, the last equality holds for $0 \leq k \leq r$ if $r \geq 4$, and either the homology group $H_{r-2}(M;\mathbb{Z})$ has no 2-torsion or $A^{r-3} = 0$.*

Clearly, the condition $A^i = 0$ for $1 \leq i \leq r-4$ is vacuous if $r = 3$ or $r = 4$. The case $r = 3$ is of particular interest.

**Corollary 1.8** *Let $M$ be a compact connected smooth manifold of dimension at least 7, whose homology group $H_2(M;\mathbb{Z})$ has no 2-torsion. For an admissible subring $A$ of $H^*(M;\mathbb{Z}/2)$, the following conditions are equivalent:*

(a) *There exists an algebraic model $(X,\varphi)$ of $M$ satisfying*

$$\varphi^*(A) \subseteq H^*_{\mathrm{alg}}(X;\mathbb{Z}/2) \quad \textit{and} \quad \varphi^*(A^k) = H^k_{\mathrm{alg}}(X;\mathbb{Z}/2) \quad \textit{for } k = 0,1,2,3.$$

(b) *$w_k(M)$ is in $A^k$ for $k = 0,1,2,3$.*

*Proof* It suffices to prove that (b) implies (a). According to Proposition 1.12($p_3$), the subring $A$ is 3-admissible, and hence, it suffices to apply Theorem 1.7 with $r = 3$. □

A much weaker version of Corollary 1.8 was proved in [45] for a spin manifold $M$ whose homology group $H_i(M;\mathbb{Z})$ has no 2-torsion for $i = 1,2$. By definition, $M$ is a spin manifold if $w_1(M) = 0$ and $w_2(M) = 0$, which automatically implies $w_3(M) = 0$ (cf. [52, p. 94]).

**Corollary 1.9** *Let $M$ be a compact connected smooth manifold of dimension at least 7, whose homology group $H_2(M;\mathbb{Z})$ has no 2-torsion. For an admissible subring $A$ of $H^*(M;\mathbb{Z}/2)$, the following conditions are equivalent:*

(a) *There exists an algebraic model $(X,\varphi)$ of $M$ satisfying*

$$\varphi^*(A^3) = H^3_{\mathrm{alg}}(X;\mathbb{Z}/2).$$

(b) *$w_1(M) \cup w_1(M) \cup w_1(M)$, $w_1(M) \cup w_2(M)$ and $w_3(M)$ are in $A^3$.*

*Proof* It is already known that (a) implies (b).

Suppose now that (b) holds. According to Proposition 1.12($p_3$), the subgroup $A^3$ of $H^3(M;\mathbb{Z}/2)$ is generated by regular cohomology classes. Hence, the subring $C$ of $H^*(M;\mathbb{Z}/2)$ generated by $A^3$ and $w_i(M)$ for $i \geq 0$ is admissible and $C^3 = A^3$. Condition (a) follows by applying Corollary 1.8 to the subring $C$. □

Theorem 1.7 with $r = 4$ takes the following form:

**Corollary 1.10** *Let $M$ be a compact connected smooth manifold of dimension at least 9, whose homology group $H_3(M;\mathbb{Z})$ has no 2-torsion. For a 4-admissible subring $A$ of $H^*(M;\mathbb{Z}/2)$, the following conditions are equivalent:*

(a) *There exists an algebraic model $(X,\varphi)$ of $M$ satisfying*

$$\varphi^*(A) \subseteq H^k_{\mathrm{alg}}(X;\mathbb{Z}/2) \quad \textit{and} \quad \varphi^*(A^k) = H^k_{\mathrm{alg}}(X;\mathbb{Z}/2) \quad \textit{for } k = 0, 1, 2, 4.$$

(b) *$w_j(M)$ is in $A^j$ for $j = 0, 1, 2, 3, 4$.*

*Moreover, $k = 3$ can be added in condition* (a) *if either the homology group $H_2(M;\mathbb{Z})$ has no 2-torsion or $A^1 = 0$.*

*Proof* Since the ring $A$ is 4-admissible, it readily follows that $\mathrm{Sq}^1(A^2) \subseteq A^3$. By Wu's formula [52, p. 94], $\mathrm{Sq}^1(w_2(M)) = w_1(M) \cup w_2(M) + w_3(M)$. Consequently, if $w_j(M)$ is in $A^j$ for $j = 1, 2$, then $w_3(M)$ is in $A^3$. If (a) holds, then $w_j(M)$ is in $A^j$ for $j = 0, 1, 2, 4,$, and hence, (b) is satisfied. According to Theorem 1.7 with $r = 4$, condition (b) implies (a). □

It is an open problem whether or not Corollary 1.10 remains true if the homology group $H_i(M;\mathbb{Z})$ has no 2-torsion for $i = 2, 3$ and the subring $A$ is admissible, but not necessarily 4-admissible. No result similar to Corollary 1.10 is available in the literature.

Under an additional assumption on $M$, Corollary 1.6 can be strengthened as follows.

**Corollary 1.11** *Let $M$ be a compact connected smooth manifold whose homology group $H_{r-1}(M;\mathbb{Z})$ has no 2-torsion for some integer $r \geq 3$ with $2r + 1 \leq \dim M$. Let $G$ be a subgroup of $H^r(M;\mathbb{Z}/2)$ generated by some regular cohomology classes and all cup products $w_{i_1}(M) \cup \cdots \cup w_{i_p}(M)$, where $i_1, \ldots i_p$ are nonnegative integers with $i_1 + \cdots + i_p = r$. If $w_i(M) = 0$ for $1 \leq i \leq r - 4$, then there exists an algebraic model $(X,\varphi)$ of $M$ satisfying $\varphi^*(G) = H_{\mathrm{alg}}(X;\mathbb{Z}/2)$.*

*Proof* The subring $A$ of $H^*(M;\mathbb{Z}/2)$ generated by $G$ and $w_j(M)$ for $j \geq 0$ is $r$-admissible, $A^i = 0$ for $1 \leq i \leq r - 4$, and $A^r = G$. Hence, it suffices to apply Theorem 1.7. □

In Corollary 1.11, the condition $w_i(M) = 0$ for $1 \leq i \leq r - 4$ is vacuous if $r = 3$ or $r = 4$, while it is equivalent to the orientability of $M$ if $r = 5$. It follows from Proposition 1.12($\mathrm{p}_3$) that Corollary 1.9 is equivalent to Corollary 1.11 with $r = 3$.

The properties of admissible rings used in the proofs of the corollaries above are contained in the following:

**Proposition 1.12** *Let $M$ be a compact connected smooth manifold. Any admissible subring $M$ of $H^*(M;\mathbb{Z}/2)$ has the following properties:*

($p_1$) *$A$ is 2-admissible.*
($p_2$) *If $\mathrm{Sq}^1(A^2) \subseteq A^3$, then there exists a 3-admissible subring $\overline{A}$ of $H^*(M;\mathbb{Z}/2)$ satisfying $A \subseteq \overline{A}$ and $A^i = \overline{A}^i$ for $i = 0, 1, 2, 3$.*
($p_3$) *If the homology group $H_2(M;\mathbb{Z})$ has no 2-torsion, then $A$ is 3-admissible and the subgroup $A^3$ of $H^3(M;\mathbb{Z}/2)$ is generated by regular cohomology classes.*

*Proof* By Künneth's theorem, each subring of $H^*(M;\mathbb{Z}/2)$ that is generated by two full subrings is also full.

The admissible subring $A$ can be written as $A = B[T]$, where $B$ is a full subring and $T$ is a regular subset of $H^*(M;\mathbb{Z}/2)$. Let $T^i := T \cap H^i(M;\mathbb{Z}/2)$ for $i \geq 0$. One has $A^0 = B^0 = H^0(M;\mathbb{Z}/2)$, the manifold $M$ being connected, and hence, it can be assumed that $T^0 = \emptyset$.

For each cohomology class $u$ in $H^1(M;\mathbb{Z}/2)$, let $\gamma_u$ be a real line bundle on $M$ with $w_1(\gamma_u) = u$. Let $F_1 := \{\gamma_u \mid u \in T^1\}$. The subring $B(F_1)$ of $H^*(M;\mathbb{Z}/2)$ generated by $B$ and $F_1(M)$ is full. Property $(p_1)$ follows since $A = B(F_1)[T \setminus T^1]$.

According to Wu's formula [52, p. 94], for each real vector bundle $\xi$ on $M$,

$$\mathrm{Sq}^1(w_2(\xi)) = w_1(\xi) \cup w_2(\xi) + w_3(\xi). \tag{$*$}$$

For each cohomology class $v$ in $W^2(M)$, let $\xi_v$ be a real vector bundle on $M$ with $w_1(\xi_v) = 0$ and $w_2(\xi_v) = v$. The admissibility of $A$ implies that $A^2$ is contained in $A^2(M) = W^2(M)$. In particular, the set $F_2 := \{\xi_v \mid v \in T^2\}$ is well defined. The subring $B(F_1, F_2)$ of $H^*(M;\mathbb{Z}/2)$ generated by $B$ and $(F_1 \cup F_2)(M)$ is full, and the subring $\overline{A} := B(F_1, F_2)[T \setminus (T^1 \cup T^2)]$ is 3-admissible. Moreover, $A \subseteq \overline{A}$ and $A^i = \overline{A}^i$ for $i = 0, 1, 2$. If $\mathrm{Sq}^1(A^2) \subseteq A^3$, then $(*)$ with $\xi = \xi_v$ implies that $w_3(\xi_v) = \mathrm{Sq}^1(v)$ is in $A^3$ for $v$ in $T^2$. Consequently, $A^3 = \overline{A}^3$. Property $(p_2)$ is proved.

Suppose now that the homology group $H_2(M;\mathbb{Z})$ has no 2-torsion. According to the universal coefficient theorem, the cohomology group $H^3(M;\mathbb{Z})$ has no 2-torsion and the reduction modulo 2 homomorphism $\rho\colon H^2(M;\mathbb{Z}) \to H^2(M;\mathbb{Z}/2)$ is surjective. For each cohomology class $z$ in $H^2(M;\mathbb{Z})$, let $\lambda_z$ be a complex line bundle on $M$ whose first Chern class is $z$. Regarding $\lambda_z$ as a rank 2 real vector bundle, one gets $w_1(\lambda_z) = 0$ and $w_2(\lambda_z) = \rho(z)$. Consequently, $W^2(M) = H^2(M;\mathbb{Z}/2)$, and it can be assumed that for each $v$ in $H^2(M;\mathbb{Z}/2)$, the vector bundle $\xi_v$ above is of rank 2. In particular, $w_j(\xi_v) = 0$ for $j \geq 3$. It follows that then $A$ is equal to the subring $\overline{A}$ constructed above, and hence, $A$ is 3-admissible. It remains to prove that $A^3$ is generated by regular cohomology classes. Each cohomology class in $H^1(M,\mathbb{Z}/2)$ is regular. Similarly, each cohomology class $v$ in $H^2(M;\mathbb{Z}/2)$ is regular since it is Poincaré dual to the homology class represented by the zero locus of an arbitrary smooth section of $\xi_v$ that is transverse to the zero section (cf. [27, Proposition 2.15]). The homomorphism $\mathrm{Sq}^1\colon H^2(M;\mathbb{Z}/2) \to H^3(M;\mathbb{Z}/2)$ is zero, the homomorphism $\rho$ being surjective [52, p. 182], and hence, $(*)$ gives $w_3(\xi) = w_1(\xi) \cup w_2(\xi)$. The proof is complete since cup product of regular cohomology classes is a regular class. □

**Convention** *Henceforth, smooth submanifolds are assumed to be closed subsets of the ambient manifold.*

## 2 Proofs and further results

The language of real algebraic geometry, as in [16], is used throughout this section. The term *real algebraic variety* designates a locally ringed space isomorphic to an algebraic subset of $\mathbb{R}^n$, for some $n$, endowed with the Zariski topology and the sheaf of real-valued regular functions (such objects are called affine real algebraic varieties in [16]). The Grassmannian $\mathbb{G}_{n,r}(\mathbb{R})$ of $r$-dimensional vector subspaces of $\mathbb{R}^n$ is a real algebraic variety in this sense [16, Theorem 3.4.4]. Morphisms between real algebraic varieties are called *regular maps*. Every real algebraic variety carries also the Euclidean topology, which is induced by the usual matric on $\mathbb{R}$. Unless explicitly stated otherwise, all topological notions relating to real algebraic varieties refer to the Euclidean topology.

A topological real vector bundle on a real algebraic variety $X$ is said to admit an *algebraic structure* if it is isomorphic to an algebraic subbundle of the trivial vector bundle on $X$ with total space $X \times \mathbb{R}^p$ for some $p$.

For any smooth manifolds $N$ and $P$, the space of smooth maps $\mathcal{C}^\infty(N, P)$ is endowed with the $\mathcal{C}^\infty$ topology [30]. The source manifold will always be assumed to be compact, and hence, the weak $\mathcal{C}^\infty$ topology coincides with the strong one. The unoriented bordism group of $P$ is denoted by $\mathfrak{N}_*(P)$. If $W$ is a nonsingular real algebraic variety, then a bordism class in $\mathfrak{N}_*(W)$ is said to be *algebraic*, provided that it can be represented by a regular map from a compact nonsingular real algebraic variety into $W$. The set $\mathfrak{N}_*^{\mathrm{alg}}(W)$ of all algebraic bordism classes in $\mathfrak{N}_*(W)$ forms a subgroup.

The main approximation theorem of real algebraic geometry, in the form most suitable for this paper, will be recalled first. It is just a reformulation of very similar results proved in [1,8,13,14,64].

**Theorem 2.1** (cf. [42, Theorem 4.4]) *Let $M$ be a compact smooth submanifold of $\mathbb{R}^n$ and let $W$ be a nonsingular real algebraic variety. Let $f\colon M \to W$ be a smooth map whose bordism class in $\mathfrak{N}_*(W)$ is algebraic. Suppose that $M$ contains a (possibly empty) subset $Z$ which is the union of finitely many nonsingular algebraic subsets of $\mathbb{R}^n$, $f|_Z\colon Z \to W$ is a regular map, and the restriction to $Z$ of the tangent bundle of $M$ admits an algebraic structure. If* $2\dim M + 1 \le n$, *then there exists a smooth embedding $e\colon M \to \mathbb{R}^n$, a nonsingular algebraic subset $X$ of $\mathbb{R}^n$, and a regular map $g\colon X \to W$ such that $X = e(M)$, $Z \subseteq X$, $e|_Z\colon Z \to \mathbb{R}^n$ is the inclusion map, $g|_Z = f|_Z$, and $g \circ \overline{e}$ is homotopic to $f$, where $\overline{e}\colon M \to X$ is the smooth diffeomorphism defined by $\overline{e}(x) = e(x)$ for all $x$ in $M$. Furthermore, given a neighborhood $\mathcal{U}$ of the inclusion map $M \hookrightarrow \mathbb{R}^n$ in the space $\mathcal{C}^\infty(M, \mathbb{R}^n)$ and a neighborhood $\mathcal{V}$ of $f$ in $\mathcal{C}^\infty(M, W)$, the objects $e$, $X$, $g$ can be chosen in such a way that $e$ is in $\mathcal{U}$ and $g \circ \overline{e}$ is in $\mathcal{V}$.*

In favorable situations, the bordism condition in Theorem 2.1 is automatically satisfied.

**Proposition 2.2** *Let $V$ and $W$ be compact nonsingular real algebraic varieties. Then:*

(i) $\mathfrak{N}_*^{\mathrm{alg}}(V) = \mathfrak{N}_*(V)$ *if and only if* $H_*^{\mathrm{alg}}(V; \mathbb{Z}/2) = H_*(V; \mathbb{Z}/2)$.

(ii) *The equality* $H_*^{\mathrm{alg}}(V \times W; \mathbb{Z}/2) = H_*(V \times W; \mathbb{Z})$ *holds, provided that* $H_*^{\mathrm{alg}}(V; \mathbb{Z}/2) = H_*(V; \mathbb{Z}/2)$ *and* $H_*^{\mathrm{alg}}(W; \mathbb{Z}/2) = H_*(W; \mathbb{Z}/2)$.

*Moreover,* $H_*^{\mathrm{alg}}(\mathbb{G}_{n,r}(\mathbb{R}); \mathbb{Z}/2) = H_*(\mathbb{G}_{n,r}(\mathbb{R}); \mathbb{Z}/2)$.

*Proof* Condition (i) is a consequence of deep results from topology (cf. [8, Lemma 2.7.1]). Condition (ii) follows from Künneth's theorem. The last assertion is a standard fact (cf. [16, Proposition 11.3.3]). □

The result that will be recalled next is used in constructions of nonalgebraic cohomology classes. For any compact nonsingular real algebraic variety $X$, let $\mathrm{Alg}^k(X)$ denote the set of all elements $u$ in $H^k(X; \mathbb{Z}/2)$ for which there exist a compact nonsingular irreducible real algebraic variety $T$, two points $t_0$ and $t_1$ in $T$ and the cohomology class $z$ in $H^k_{\mathrm{alg}}(X \times T; \mathbb{Z}/2)$ such that

$$u = i_{t_1}^*(z) - i_{t_0}^*(z),$$

where $i_t \colon X \to X \times T$ is defined by $i_t(x) = (x, t)$ for $t \in T$ and $x \in X$. An equivalent description of $\mathrm{Alg}^k(X)$, which immediately implies that $\mathrm{Alg}^k(X)$ is a subgroup of $H^k_{\mathrm{alg}}(X; \mathbb{Z}/2)$, is given in [38,40]. The groups $H^k_{\mathrm{alg}}(-; \mathbb{Z}/2)$ and $\mathrm{Alg}^k(-)$ have the expected functorial property. If $f \colon X \to Y$ is a regular map between compact nonsingular real algebraic varieties, then the induced homomorphism $f^* \colon H^*(Y; \mathbb{Z}/2) \to H^*(X; \mathbb{Z}/2)$ satisfies

$$f^*(H^k_{\mathrm{alg}}(Y; \mathbb{Z}/2)) \subseteq H^k_{\mathrm{alg}}(X; \mathbb{Z}/2) \quad \text{and} \quad f^*(\mathrm{Alg}^k(Y)) \subseteq \mathrm{Alg}^k(X)$$

(cf. [27, Section 5] or [4,15] for the former inclusion and [40] for the latter).

*Example 2.3* Let $\Sigma$ be an irreducible nonsingular real algebraic variety with precisely two connected components $\Sigma_0$ and $\Sigma_1$, each diffeomorphic to the unit circle. For example, one can take

$$\Sigma = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^4 - 4x_1^2 + x_2^2 + 1 = 0\}.$$

Let $z$ be the cohomology class in $H^1(\Sigma \times \Sigma; \mathbb{Z}/2)$ that is Poincaré dual to the homology class in $H_1(\Sigma \times \Sigma; \mathbb{Z}/2)$ represented by the diagonal of $\Sigma \times \Sigma$. For any point $t$ in $\Sigma$, let $i_t \colon \Sigma \to \Sigma \times \Sigma$ be defined by $i_t(x) = (x, t)$ for all $x$ in $\Sigma$. The cohomology class $i_t^*(z)$ in $H^1(\Sigma; \mathbb{Z}/2)$ is Poincaré dual to the homology class in $H_1(\Sigma; \mathbb{Z}/2)$ represented by the point $t$. Let $t_j$ be a point in $\Sigma_j$ for $j = 0, 1$. The cohomology class $u := i_{t_1}^*(z) - i_{t_0}^*(z)$ is in $\mathrm{Alg}^1(\Sigma)$. If $\sigma \colon \Sigma_0 \hookrightarrow \Sigma$ is the inclusion map, then $\sigma^*(u)$ generates $H^1(\Sigma_0; \mathbb{Z}/2) \cong \mathbb{Z}/2$ and hence

$$H^1(\Sigma_0; \mathbb{Z}/2) = \sigma^*(H^1(\Sigma; \mathbb{Z}/2)) = \sigma^*(\mathrm{Alg}^1(\Sigma)).$$

Consequently, the functoriality of $\mathrm{Alg}^1(-)$ implies that

$$r^*(H^1(\Sigma; \mathbb{Z}/2)) \subseteq \mathrm{Alg}^1(Y)$$

for every nonsingular real algebraic variety $Y$ and every regular map $r \colon Y \to \Sigma$ with $r(Y) \subseteq \Sigma_0$.

As usual, the Kronecker index (scalar product) of cohomology and homology classes will be denoted by $\langle -, - \rangle$. For any $m$-dimensional compact smooth manifold $M$, let $[M]$ denote its fundamental class in $H_m(M; \mathbb{Z}/2)$.

**Theorem 2.4** (cf. [38, Theorem 2.1]) *Let $X$ be a compact nonsingular real algebraic variety. Then, $\langle u \cup v, [X] \rangle = 0$ for all $u$ in $H^k_{\mathrm{alg}}(X; \mathbb{Z}/2)$ and $v$ in $\mathrm{Alg}^l(X)$, where $k + l = \dim X$.*

If $K$ is a $k$-dimensional smooth submanifold of $M$, let $[K]_M$ denote the homology class in $H_k(M; \mathbb{Z}/2)$ represented by $K$, that is, $[K]_M := \kappa_*([K])$, where $\kappa : K \hookrightarrow M$ is the inclusion map. The unit 1-sphere and the unit 1-disk will be denoted by $\mathbb{S}^1$ and $\mathbb{D}^1$, respectively.

The following technical result will be very useful.

**Lemma 2.5** *Let $L$ be a $(k+1)$-dimensional compact smooth submanifold of $\mathbb{R}^n$ and let $K$ be a $k$-dimensional smooth submanifold of $L$ such that there is a smooth diffeomorphism $\theta \colon K \times \mathbb{S}^1 \to L$ satisfying $\theta(K \times \{z_0\}) = K$ for some point $z_0$ in $\mathbb{S}^1$. Let $f \colon L \to V$ be a smooth map into a nonsingular real algebraic variety $V$. Let $\mathcal{U}$ be a neighborhood of the inclusion map $L \hookrightarrow \mathbb{R}^n$ in the space $\mathcal{C}^\infty(L, \mathbb{R}^n)$ and let $\mathcal{V}$ be a neighborhood of $f$ in $\mathcal{C}^\infty(L, V)$. Assume that $2k + 3 \le n$, the map $f \circ \theta \colon K \times \mathbb{S}^1 \to V$ has a continuous extension $K \times \mathbb{D}^1 \to V$, and the bordism class of the map $f|_K \colon K \to V$ in the group $\mathfrak{N}_*(V)$ is 0. Then, there exists a smooth embedding $\varepsilon \colon L \to \mathbb{R}^n$, a nonsingular algebraic subset $Y$ of $\mathbb{R}^n$, and a regular map $g \colon Y \to V$ such that $Y = \varepsilon(L)$, $\varepsilon$ is in $\mathcal{U}$, $g \circ \overline{\varepsilon}$ is in $\mathcal{V}$, and*

$$H^k_{\mathrm{alg}}(Y; \mathbb{Z}/2) \subseteq \{w \in H^k(Y; \mathbb{Z}/2) \mid \langle w, \overline{\varepsilon}_*([K]_L) \rangle = 0\},$$

*where $\overline{\varepsilon} \colon L \to Y$ is the smooth diffeomorphism determined by $\varepsilon$.*

*Proof* Let $\Sigma$ be as in Example 2.3, and let $h_0 \colon \mathbb{S}^1 \to \Sigma$ be a smooth embedding onto $\Sigma_0$. If $f_0 \colon K \to V$ is defined by $f_0(x) = f(\theta(x, z_0))$ for all $x$ in $K$, then the bordism class of $f_0 \times h_0 \colon K \times \mathbb{S}^1 \to V \times \Sigma$ in the group $\mathfrak{N}_*(V \times \Sigma)$ is 0. Indeed, this assertion follows since the bordism classes of $f_0 \colon K \to V$ and $f|_K \colon K \to V$ in $\mathfrak{N}_*(V)$ are equal, and the latter class is 0 by assumption.

If $F \colon K \times \mathbb{D}^1 \to V$ is a continuous extension of $f \circ \theta \colon K \times \mathbb{S}^1 \to V$, then the map $H \colon K \times \mathbb{S}^1 \times [0, 1] \to V$,

$$H(x, z, t) = F(x, (1 - t)z + tz_0) \text{ for } (x, z, t) \text{ in } K \times \mathbb{S}^1 \times [0, 1],$$

is a homotopy from $f \circ \theta$ to $f_0 \circ \pi$, where $\pi : K \times \mathbb{S}^1 \to K$ is the canonical projection. Hence, if $\rho \colon K \times \mathbb{S}^1 \to \mathbb{S}^1$ is the canonical projection and $h := h_0 \circ \rho \circ \theta^{-1}$, the map

$$(f, h) \circ \theta = (f \circ \theta, h \circ \theta) \colon K \times \mathbb{S}^1 \to V \times \Sigma$$

is homotopic to

$$(f_0 \circ \pi, h_0 \circ \rho) = f_0 \times h_0 \colon K \times \mathbb{S}^1 \to V \times \Sigma.$$

Consequently, the bordism class of $(f, h) \colon L \to V \times \Sigma$ in $\mathfrak{N}_*(V \times \Sigma)$ is 0.

By Theorem 2.1 (with $M = L$, $Z = \emptyset$, and $W = V \times \Sigma$), there exist a smooth embedding $\varepsilon \colon L \to \mathbb{R}^n$, a nonsingular algebraic subset $Y$ of $\mathbb{R}^n$, and a regular map $(h, r) \colon Y \to V \times \Sigma$ such that $Y = \varepsilon(L)$, $\varepsilon$ is in $\mathcal{U}$, and $(g, r) \circ \overline{\varepsilon}$ is close to $(f, h)$ in $\mathcal{C}^\infty(L, V \times \Sigma)$, where $\overline{\varepsilon} \colon L \to Y$ is the smooth diffeomorphism determined by $\varepsilon$. In particular, $g \circ \overline{\varepsilon}$ is in $\mathcal{V}$, and $r$ is homotopic to $h \circ \overline{\varepsilon}^{-1}$. The proof can be completed as follows. Let $v$ be the cohomology class in $H^1(\Sigma; \mathbb{Z}/2)$ that is Poincaré dual to the homology class represented by the point $y_0 := h_0(z_0)$. Since $y_0$ is a regular value of $h \circ \overline{\varepsilon}^{-1}$ and $\overline{\varepsilon}(K) = (h \circ \overline{\varepsilon}^{-1})^{-1}(y_0)$, it follows that the cohomology class $(h \circ \overline{\varepsilon}^{-1})^*(v)$ is Poincaré dual to the homology class $[\overline{\varepsilon}(K)]_Y = \overline{\varepsilon}_*([K]_L)$ (cf. [27, Proposition 2.15]). Consequently, $r^*(v)$ is Poincaré dual to $\overline{\varepsilon}_*([K]_L)$, the maps $h \circ \overline{\varepsilon}^{-1}$ and $r$ being homotopic. Thus, $r^*(v) \cap [Y] = \overline{\varepsilon}_*([K]_L)$ and hence for every cohomology class $w$ in $H^1(Y; \mathbb{Z}/2)$,

$$\langle w, \overline{\varepsilon}_*([K]_L)\rangle = \langle w, r^*(v) \cap [Y]\rangle = \langle w \cup r^*(v), [Y]\rangle.$$

Since $r$ is a regular map and $r(Y) \subseteq \Sigma_0$, Example 2.3 implies that $r^*(v)$ is in $\mathrm{Alg}^1(Y)$. Hence, according to Theorem 2.4,

$$H^k_{\mathrm{alg}}(Y; \mathbb{Z}/2) \subseteq \{w \in H^k(Y; \mathbb{Z}/2) | \langle w, \overline{\varepsilon}_*([K]_L)\rangle = 0\},$$

as required. □

The ability to verify the bordism hypothesis in Lemma 2.5 is essential for applications. This often requires the following deep result from differential topology.

**Theorem 2.6** (cf. [29, (17.3)]) *Let $f \colon K \to P$ be a smooth map between compact smooth manifolds. The bordism class of $f$ in the group $\mathfrak{N}_*(P)$ is 0 if and only if for every nonnegative integer $q$ and every cohomology class $u$ in $H^q(P; \mathbb{Z}/2)$,*

$$\langle w_{i_1}(K) \cup \cdots \cup w_{i_p}(K) \cup f^*(u), [K]\rangle = 0$$

*for all nonnegative integers $i_1, \ldots, i_p$ with $i_1 + \cdots + i_p = k - q$, where $k = \dim K$.*

Henceforth, the following notion will play a crucial role.

**Definition 2.7** Given a compact smooth manifold $M$ and a subring $A$ of $H^*(M; \mathbb{Z}/2)$, a smooth submanifold $K$ of $M$ is said to be adapted to $A$ if for every nonnegative integer $q$ and every cohomology class $u$ in $A^q$,

$$\langle w_{i_1}(K) \cup \cdots \cup w_{i_p}(K) \cup \kappa^*(u), [K]\rangle = 0$$

for all nonnegative integers $i_1, \ldots, i_p$ with $i_1 + \ldots + i_p = k - q$, where $k = \dim K$ and $\kappa \colon K \hookrightarrow M$ is the inclusion map.

For any smooth manifold $N$, let $\tau_N$ denote its tangent bundle.

**Lemma 2.8** *Let $M$ be a compact smooth manifold and let $K$ be a connected smooth submanifold of $M$ of positive dimension $k$, with $2k+1 \leq \dim M$. If $K$ is adapted to a subring $A$ of $H^*(M; \mathbb{Z}/2)$ containing the Stiefel–Whitney classes $w_i(M)$ for $0 \leq i \leq k$, then the normal bundle of $K$ in $M$ splits off a trivial vector bundle of rank 2.*

*Proof* If $2k+2 \leq \dim M$, then the assertion is true without any additional assumptions on $K$.

Suppose now that $2k+1 = \dim M$. It suffices to prove that the normal bundle $\nu$ of $K$ in $M$ has two continuous sections that are linearly independent at each point of $K$. Since $\operatorname{rank} \nu = k+1$ and $\dim K = k$, the only obstruction to the existence of such sections is an element $W_k(\nu)$ in the cohomology group $H^k(X; \Gamma)$, where $\Gamma$ is a local system of coefficients with fiber $\mathbb{Z}$ or $\mathbb{Z}/2$ (cf. [52, p. 140] and [60, pp. 190, 191]).

If $k$ is even, then $\Gamma$ is isomorphic to the constant local system $\mathbb{Z}/2$, and $W_k(\nu)$ can be identified with $w_k(\nu)$ (cf. [52, p. 143]).

If $k$ is odd, then the local system $\Gamma$ has fiber $\mathbb{Z}$. The group $H^k(K; \Gamma)$ is isomorphic either to $\mathbb{Z}$ or $\mathbb{Z}/2$. Indeed, the Poincaré duality gives an isomorphism between $H^k(K; \Gamma)$ and the 0th homology group of $K$ with a suitable local system of coefficients with fiber $\mathbb{Z}$. The 0th homology group of $K$ with an arbitrary local system of coefficients with fiber $\mathbb{Z}$ is isomorphic either to $\mathbb{Z}$ or $\mathbb{Z}/2$. If the group $H^k(K; \Gamma)$ is infinite cyclic, then $W_k(\nu) = 0$ since $W_k(\nu)$ is an element of order 2 (cf. [60, Theorem 38.11]). If $H^k(K; \Gamma)$ is isomorphic to $\mathbb{Z}/2$, then the reduction modulo 2 homomorphism $\rho\colon H^k(K; \Gamma) \to H^k(K; \mathbb{Z}/2)$ is an isomorphism. According to [52, Theorem 12.1], $\rho(W_k(\nu)) = w_k(\nu)$.

In conclusion, $W_k(\nu) = 0$ if $w_k(\nu) = 0$. It remains to prove the equality $w_k(\nu) = 0$. If $\kappa\colon K \hookrightarrow M$ is the inclusion map, then the vector bundles $\tau_K \oplus \nu$ and $\kappa^*(\tau_M)$ are isomorphic, and hence, one gets $w(K) \cup w(\nu) = \kappa^*(w(M))$ for the total Stiefel–Whitney classes. The last equality implies that $w_k(\nu)$ belongs to the subring of $H^*(K; \mathbb{Z}/2)$ generated by $w_i(K)$ and $\kappa^*(w_i(M))$ for $0 \leq i \leq k$. Consequently, $\langle w_k(\nu), [K] \rangle = 0$ since $K$ is adapted to $A$ and $w_i(M)$ is in $A^i$ for $0 \leq i \leq k$. Thus, $w_k(\nu) = 0$, the manifold $K$ being connected. □

The next lemma is included for the sake of completeness. If $M$ is a compact smooth manifold and $N$ is a smooth submanifold of $M$ of codimension $k$, let $[N]^M$ denote the cohomology class in $H^k(M; \mathbb{Z}/2)$ that is Poincaré dual to the homology class $[N]_M$ represented by $N$. That is, $[N]^M \cap [M] = [N]_M$, where $\cap$ denotes the cap product.

**Lemma 2.9** *Let $M$ be a compact smooth manifold of dimension $m$. Let $K_1, \ldots, K_p$ be pairwise disjoint connected smooth submanifolds of $M$ of dimension $k$, where $1 \leq k \leq m-1$. Let $N$ be a smooth submanifold of $M$ of codimension $k$. If*

$$\langle [N]^M, [K_l]_M \rangle = 0 \quad \textit{for } 1 \leq l \leq p,$$

*then there exists a smooth submanifold $N'$ of $M$ of codimension $k$ such that $[N']^M = [N]^M$ and $K_l \cap N' = \emptyset$ for $1 \leq l \leq p$.*

*Proof* Arguing by induction on the number of submanifolds $K_l$ suppose that $j$ is an integer satisfying $0 \leq j \leq p-1$, and $N_j$ is a smooth submanifold of $M$ with

$[N_j]^M = [N]^M$ and $K_l \cap N_j = \emptyset$ for $1 \le l \le j$ (the last condition is vacuous if $j = 0$). It suffices to prove the existence of a smooth submanifold $N_{j+1}$ of $M$ such that $[N_{j+1}]^M = [N]^M$ and $K_l \cap N_{j+1} = \emptyset$ for $1 \le l \le j+1$. The submanifold $N_j$ can be assumed to be transverse to $K_l$ for $1 \le l \le p$. Since

$$\begin{aligned}\langle [N_j]^M \cup [K_{j+1}]^M, [M]\rangle &= \langle [N_j]^M, [K_{j+1}]^M \cap [M]\rangle \\ &= \langle [N_j]^M, [K_{j+1}]_M\rangle = 0,\end{aligned}$$

the modulo 2 intersection number of $K_{j+1}$ and $N_j$ in $M$ is 0, and hence, the set $K_{j+1} \cap N_j$ is either empty or consists of $2r$ points for some positive integer $r$. In the former case, it suffices to set $N_{j+1} := N_j$. In the latter case, let $x$ and $y$ be two points in $K_{j+1} \cap N_j$ that can be joined by a smooth arc $C$ in $K_{j+1}$ satisfying $C \cap N_j = \{x, y\}$. The restriction to $C$ of the normal bundle of $K_{j+1}$ in $M$ is trivial, and hence making use of a thin $(m-k)$-dimensional tube along $C$, one can construct a smooth submanifold $N_j(x, y)$ of $M$ with $[N_j(x, y)]^M = [N_j]^M$, $K_l \cap N_j(x, y) = \emptyset$ for $1 \le l \le j$, and $K_{j+1} \cap N_j(x, y) = (K_{j+1} \cap N_j) \setminus \{x, y\}$. By repeating this procedure $r$ times, one obtains a smooth submanifold $N_{j+1}$ of $M$ having the required properties. □

For any subring $A$ of $H^*(M; \mathbb{Z}/2)$, let

$$A_k := \{\alpha \in H_k(M; \mathbb{Z}/2) \mid \langle u, \alpha\rangle = 0 \quad \text{for all } u \in A^k\}.$$

**Theorem 2.10** *Let $M$ be a compact connected smooth manifold and let $r$ be a positive integer with $2r + 1 \le \dim M$. Let $A$ be an $r$-admissible subring of $H^*(M; \mathbb{Z}/2)$ and let $\Delta$ be the set of all integers $k$ such that $1 \le k \le r$ and the group $A_k$ is generated by homology classes of the form $[K]_M$, where $K$ is a $k$-dimensional connected smooth submanifold of $M$ adapted to $A$. If $w_i(M)$ is in $A^i$ for $0 \le i \le r$, then there exists an algebraic model $(X, \varphi)$ of $M$ satisfying*

$$\varphi^*(A) \subseteq H^*_{\mathrm{alg}}(X; \mathbb{Z}/2) \quad \textit{and} \quad \varphi^*(A^k) = H^k_{\mathrm{alg}}(X; \mathbb{Z}/2) \quad \textit{for } k \textit{ in } \{0\} \cup \Delta.$$

*Proof* The subring $A$ can be written as $A = B[T]$, where $B$ is a full subring and $T$ is a regular subset of $H^*(M; \mathbb{Z}/2)$, with $T$ disjoint from $H^c(M; \mathbb{Z}/2)$ for $0 \le c \le r-1$. By definition,

$$B = h^*(H^*(W; \mathbb{Z}/2)), \tag{1}$$

where $h\colon M \to W$ is a smooth map into a compact smooth manifold $W$ with $A(W) = H^*(W; \mathbb{Z}/2)$. In view of the last equality, the whole ring $H^*(W; \mathbb{Z}/2)$ is algebraically realizable (cf. Sect. 1), and hence, it can be assumed that $W$ is a nonsingular real algebraic variety satisfying

$$H^*_{\mathrm{alg}}(W; \mathbb{Z}/2) = H^*(W; \mathbb{Z}/2). \tag{2}$$

Let $m := \dim M$. If $d$ is a sufficiently large integer and $G := \mathbb{G}_{d,m}(\mathbb{R})$, then there exists a smooth classifying map $g\colon M \to G$ for the tangent bundle $\tau_M$ of $M$, that is,

$\tau_M$ is isomorphic to the pullback $g^*\gamma$ of the universal vector bundle $\gamma$ on $G$. Hence, the subring $g^*(H^*(H;\mathbb{Z}/2))$ of $H^*(M;\mathbb{Z}/2)$ is generated by $w_i(M)$ for $i \geq 0$. The smooth map $(g,h)\colon M \to G \times W$ plays a crucial role. Set

$$D := (g,h)^*(H^*(G \times W; \mathbb{Z}/2)) \text{ and } C := D[T]. \tag{3}$$

Since $w_i(M)$ is in $A^i$ for $0 \leq i \leq r$, by (1) and Künneth's theorem, the subring $C$ of $H^*(M;\mathbb{Z}/2)$ satisfies

$$A \subseteq C \quad \text{and} \quad A^i = C^i,\ A_i = C_i \quad \text{for } 0 \leq i \leq r. \tag{4}$$

By (2) and Proposition 2.2,

$$\mathfrak{N}_*^{\mathrm{alg}}(W) = \mathfrak{N}_*(W) \quad \text{and} \quad \mathfrak{N}_*^{\mathrm{alg}}(G \times W) = \mathfrak{N}_*(G \times W). \tag{5}$$

In view of (4), if $p$ is a sufficiently large integer, then for each integer $k$ in $\Delta$, there exist $k$-dimensional connected smooth submanifolds $K_{k1}, \dots, K_{kp}$ of $M$ such that

$$\text{each } K_{kl} \text{ is adapted to } C, \tag{6}$$
$$[K_{k1}]_M, \dots, [K_{kp}]_M \text{ generate } C_k = A_k. \tag{7}$$

By (6) and Lemma 2.8,

$$\begin{gathered}\text{the normal bundle of each } K_{kl} \text{ in } M \\ \text{splits off a trivial vector bundle of rank 2.}\end{gathered} \tag{8}$$

If $\kappa_{kl}\colon K_{kl} \hookrightarrow M$ is the inclusion map, the restriction map $(g,h)|_{K_{kl}}\colon K_{kl} \to G \times W$ can be written as $(g,h)|_{K_{kl}} = (g,h) \circ \kappa_{kl}$, and hence

$$\begin{aligned}&((g,h)|_{K_{kl}})^*(H^*(G \times W; \mathbb{Z}/2)) = \\ &\qquad \kappa_{kl}^*((g,h)^*(H^*(G \times W; \mathbb{Z}/2))) \subseteq \kappa_{kl}^*(C),\end{aligned}$$

where the inclusion follows from (3). Consequently, by (6) and Theorem 2.6,

$$\text{the bordism class of } (g,h)|_{K_{kl}}\colon K_{kl} \to G \times W \text{ in } \mathfrak{N}_*(G \times W) \text{ is } 0. \tag{9}$$

Let $N_1, \dots, N_q$ be smooth submanifolds of $M$ such that

$$T = \{[N_1]^M, \dots, [N_q]^M\} \quad \text{and} \quad \operatorname{codim}_M N_j \geq r \quad \text{for } 1 \leq j \leq q, \tag{10}$$

and let

$$N := N_1 \cup \cdots \cup N_q.$$

The collection of smooth submanifolds of $M$ consisting of all $K_{kl}$ and all $N_j$ can be assumed to be in general position. In particular, the $K_{kl}$ are pairwise disjoint since $2r + 1 \leq m$. Similarly, each $K_{kl}$ with $1 \leq k \leq r - 1$ is disjoint from $N$ since $\operatorname{codim}_M N_j \geq r$ for $1 \leq j \leq q$. Moreover, according to Lemma 2.9, the $N_j$ can be chosen in such a way that $K_{kl} \cap N = \emptyset$ for $k \in \Delta$ and $1 \leq l \leq p$.

One can assume that $M$ is smoothly embedded in $\mathbb{R}^n$ for some $n \geq 2m + 1$. Since (5) holds, according to [13, Theorem 4, Remark 8], it can be assumed that

$$M \text{ is a nonsingular algebraic subset of } \mathbb{R}^n, \tag{11}$$

$$N_1, \ldots, N_q \text{ are nonsingular algebraic subsets of } M, \tag{12}$$

$$(g, h)\colon M \to G \times W \text{ is a regular map.} \tag{13}$$

Let $U_{kl}$ be a tubular neighborhood of $K_{kl}$ in $M$. The $U_{kl}$ can be chosen to be pairwise disjoint and disjoint from $N$. In view of (8), one can find a smooth embedding $\eta_{kl}\colon K_{kl} \times \mathbb{D}^1 \to U_{kl}$ such that if $L_{kl} := \eta_{kl}(K_{kl} \times \mathbb{S}^1)$ and $\theta_{kl}\colon K_{kl} \times \mathbb{S}^1 \to L_{kl}$ is the smooth diffeomorphism determined by $\eta_{kl}$, then $K_{kl} = \theta_{kl}(K_{kl} \times \{z_0\})$ for some point $z_0$ in $\mathbb{S}^1$. The smooth map $((g, h)|_{L_{kl}}) \circ \theta_{kl}\colon K_{kl} \times \mathbb{S}^1 \to G \times W$ is a restriction of the smooth map $(g, h) \circ \eta_{kl}\colon K_{kl} \times \mathbb{D}^1 \to G \times W$. Hence, by (9) and Lemma 2.5 (with $K = K_{kl}$, $L = L_{kl}$ and $f = (g, h)|_{L_{kl}}$), there exist a smooth embedding $\varepsilon_{kl}\colon L_{kl} \to \mathbb{R}^n$, a nonsingular algebraic subset $Y_{kl}$ of $\mathbb{R}^n$, and a regular map $(g_{kl}, h_{kl})\colon Y_{kl} \to G \times W$ such that $Y_{kl} = \varepsilon_{kl}(L_{kl})$, $\varepsilon_{kl}$ is close to the inclusion map $L_{kl} \hookrightarrow \mathbb{R}^n$ in the space $\mathcal{C}^\infty(L_{kl}, \mathbb{R}^n)$, $(g_{kl}, h_{kl}) \circ \overline{\varepsilon}_{kl}$ is close to $(g, h)|_{L_{kl}}$ in $\mathcal{C}^\infty(L_{kl}, G \times W)$, and

$$H^k_{\mathrm{alg}}(Y_{kl}; \mathbb{Z}/2) \subseteq \{w \in H^k(Y_{kl}; \mathbb{Z}/2) | \langle w, \overline{\varepsilon}_{kl*}([K_{kl}]_{L_{kl}})\rangle = 0\}, \tag{14}$$

where $\overline{\varepsilon}_{kl}\colon L_{kl} \to Y_{kl}$ is the smooth diffeomorphism determined by $\varepsilon_{kl}$. If each $(g_{kl}, h_{kl}) \circ \overline{\varepsilon}_{kl}$ is sufficiently close to $(g, h)|_{L_{kl}}$, then one can find a smooth map $(g', h')\colon M \to G \times W$ that is homotopic to $(g, h)\colon M \to G \times W$ and satisfies

$$\begin{aligned}(g', h')|_N = (g, h)|_N \quad &\text{and} \\ &(g', h')|_{L_{kl}} = (g_{kl}, h_{kl}) \circ \overline{\varepsilon}_{kl} \text{for} k \in \Delta \quad \text{and} \quad 1 \leq l \leq p.\end{aligned} \tag{15}$$

If each $\varepsilon_{kl}$ is sufficiently close to the inclusion map $L_{kl} \hookrightarrow \mathbb{R}^n$, then

$$\text{the } Y_{kl} \text{ are pairwise disjoint and disjoint from } N, \tag{16}$$

and hence, there exists a smooth embedding $\varepsilon\colon M \to \mathbb{R}^n$ for which $\varepsilon|_{L_{kl}} = \varepsilon_{kl}$ and $\varepsilon|_N$ is the inclusion map $N \hookrightarrow \mathbb{R}^n$. Let $\overline{M} := \varepsilon(M)$ and let $\overline{\varepsilon}\colon M \to \overline{M}$ be the smooth diffeomorphism determined by $\varepsilon$. The smooth map $(\overline{g}, \overline{h}) := (g', h') \circ \overline{\varepsilon}^{-1}\colon \overline{M} \to G \times W$ satisfies $(\overline{g}, \overline{h})|_{Y_{kl}} = (g_{kl}, h_{kl})$ and $(\overline{g}, \overline{h})|_N = (g', h')|_N$. Moreover, the algebraic subset

$$Z := N \cup \bigcup_{k,l} Y_{kl}$$

of $\mathbb{R}^n$ is contained in $\overline{M}$, and by (12), (13), (15) and (16),

$$(\overline{g}, \overline{h})|_Z \colon Z \to G \times W \text{ is a regular map.} \tag{17}$$

Since $g \colon M \to G$ is a classifying map for $\tau_M$ and $g' \colon M \to G$ is homotopic to $g$, it follows that $\overline{g} = g' \circ \overline{\varepsilon}^{-1} \colon \overline{M} \to G$ is a classifying map for $\tau_{\overline{M}}$. Consequently, the regular map $\overline{g}|_Z \colon Z \to G$ is a classifying map for $\tau_{\overline{M}}|_Z$ and hence

$$\tau_{\overline{M}}|_Z \text{ admits an algebraic structure} \tag{18}$$

(cf. [16, Theorem 12.1.7]). In view of (5), (17) and (18), Theorem 2.1 can be applied to $\overline{h} \colon \overline{M} \to W$ and $Z \subseteq \overline{M}$. Therefore, there exist a smooth embedding $e \colon \overline{M} \to \mathbb{R}^n$, a nonsingular algebraic subset $X$ of $\mathbb{R}^n$ and a regular map $\lambda \colon X \to W$ such that $X = e(\overline{M})$, $Z \subseteq X$, $e(x) = x$ for all $x$ in $Z$, and $\lambda \circ \overline{e}$ is homotopic to $\overline{h}$, where $\overline{e} \colon \overline{M} \to X$ is the smooth diffeomorphism determined by $e$. The map $\varphi := \overline{\varepsilon}^{-1} \circ \overline{e}^{-1} \colon X \to M$ is a smooth diffeomorphism, and hence, $(X, \varphi)$ is an algebraic model of $M$.

By construction, $\lambda$ is homotopic to $\overline{h} \circ \overline{e}^{-1} = h' \circ \varphi$ and $h'$ is homotopic to $h$. Consequently, $\lambda$ is homotopic to $h \circ \varphi$, and hence,

$$\lambda^*(H^*(W; \mathbb{Z}/2)) = \varphi^*(h^*(H^*(W; \mathbb{Z}/2))) = \varphi^*(B),$$

where the last equality follows from (1). This implies that

$$\varphi^*(B) \subseteq H^*_{\mathrm{alg}}(X; \mathbb{Z}/2)$$

since (2) holds and $\lambda \colon X \to W$ is a regular map. The diffeomorphism $\varphi \colon X \to M$ satisfies $\varphi(x) = x$ for all $x$ in $N$, which gives $\varphi^*([N_j]^M) = [N_j]^X$ for $1 \leq j \leq q$. Thus, $\varphi^*([N_j]^M)$ belongs to $H^*_{\mathrm{alg}}(X; \mathbb{Z}/2)$, each $N_j$ being an algebraic subset of $X$. By (10),

$$\varphi^*(T) \subseteq H^*_{\mathrm{alg}}(X; \mathbb{Z}/2).$$

The last two inclusions imply that

$$\varphi^*(A) \subseteq H^*_{\mathrm{alg}}(X; \mathbb{Z}/2).$$

Since $M$ is connected, one has $A^0 = H^0(M; \mathbb{Z}/2)$ and

$$\varphi^*(A^0) = H^0(X; \mathbb{Z}/2) = H^0_{\mathrm{alg}}(X; \mathbb{Z}/2).$$

It remains to prove that if $u$ is a cohomology class in $H^k(M; \mathbb{Z}/2) \setminus A^k$ for some $k \in \Delta$, then $\varphi^*(u)$ is not in $H^k_{\mathrm{alg}}(X; \mathbb{Z}/2)$. Let $\delta_{kl} \colon Y_{kl} \hookrightarrow X$ be the inclusion map. The composite map

$$\varphi \circ \delta_{kl} \circ \overline{\varepsilon}_{kl} = \overline{\varepsilon}^{-1} \circ \overline{e}^{-1} \circ \delta_{kl} \circ \overline{\varepsilon}_{kl} \colon L_{kl} \to M$$

is the inclusion map $L_{kl} \hookrightarrow M$, and hence,

$$\langle \delta_{kl}^*(\varphi^*(u)), \overline{\varepsilon}_{kl*}([K_{kl}]_{L_{kl}})\rangle = \langle u, (\varphi \circ \delta_{kl} \circ \overline{\varepsilon}_{kl})_*([K_{kl}]_{L_{kl}})\rangle = \langle u, [K_{kl}]_M\rangle.$$

Since $u$ is not in $A^k$, condition (7) implies the existence of $l$ with $\langle u, [K_{kl}]_M\rangle \neq 0$. For this $l$, according to (14), $\delta_{kl}^*(\varphi^*(u))$ is not in $H^k_{\mathrm{alg}}(Y_{kl}; \mathbb{Z}/2)$. Consequently, $\varphi^*(u)$ is not in $H^k_{\mathrm{alg}}(X; \mathbb{Z}/2)$, the map $\delta_{kl}$ being regular. The proof is complete. □

Recall that a compact smooth manifold is said to be a *boundary* if it is diffeomorphic to the boundary of a compact smooth manifold with boundary.

Let $M$ be a compact connected smooth manifold and let $K$ be a smooth submanifold of $M$. If $K$ is adapted to a subring of $H^*(M; \mathbb{Z}/2)$, then the Stiefel–Whitney numbers of $K$ are all 0, and hence, $K$ is a boundary [62]. Conversely, if $K$ is a boundary, then its Stiefel–Whitney numbers are all 0, and hence, $K$ is adapted to the subring $H^0(M; \mathbb{Z}/2)$ of $H^*(M; \mathbb{Z}/2)$. The last observation can be generalized. This is done in the following two lemmas, in which notation $M$ and $K$ is preserved, and $k := \dim K$ is assumed to be positive. Moreover, $A$ denotes a subring of $H^*(M; \mathbb{Z}/2)$.

**Lemma 2.11** *Assume that the submanifold $K$ is a boundary and the cohomology class $[K]_M$ belongs to $A_k$. Then, $K$ is adapted to $A$ if one of the following two conditions is satisfied:*

(c$_1$) $1 \leq k \leq 2$.
(c$_2$) $k \geq 3$, $\mathrm{Sq}^1(A^{k-1}) \subseteq A^k$, *and* $A^i = 0$ *for* $1 \leq i \leq k-2$.

*Proof* By Wu's theorem [52, Theorem 11.14], the first Wu class of $K$ is equal to $w_1(K)$. In particular,

$$\mathrm{Sq}^1(a) = w_1(K) \cup a \quad \text{for all } a \text{ in } H^{k-1}(K; \mathbb{Z}/2). \tag{1}$$

Let $\kappa\colon K \hookrightarrow M$ be the inclusion map. Since $[K]_M$ is in $A_k$, for every cohomology class $z$ in $A^k$,

$$\langle \kappa^*(z), [K]\rangle = \langle z, \kappa_*([K])\rangle = \langle z, [K]_M\rangle = 0. \tag{2}$$

According to (1), for every cohomology class $v$ in $A^{k-1}$,

$$w_1(K) \cup \kappa^*(v) = \mathrm{Sq}^1(\kappa^*(v)) = \kappa^*(\mathrm{Sq}^1(v)).$$

Therefore, the inclusion $\mathrm{Sq}^1(A^{k-1}) \subseteq A^k$ (which is automatically satisfied if $1 \leq k \leq 2$) and (2) give

$$\langle w_1(K) \cup \kappa^*(v), [K]\rangle = 0. \tag{3}$$

Since $K$ is a boundary, the Stiefel–Whitney numbers of $K$ are all 0. Consequently, in view of (2) and (3), the submanifold $K$ is adapted to $A$, provided that either (c$_1$) or (c$_2$) is satisfied. □

**Lemma 2.12** *Assume that the submanifold $K$ is a boundary and the homology class $[K]_M$ belongs to $A_k$. Moreover, assume that $K$ is orientable. Then, $K$ is adapted to $A$ if one of the following two conditions is satisfied:*

(c$_1$) $1 \le k \le 4$.
(c$_2$) $k \ge 5$, $\mathrm{Sq}^2(A^{k-2}) \subseteq A^k$, $\mathrm{Sq}^2(\mathrm{Sq}^1(A^{k-3})) \subseteq A^k$, *and* $A^i = 0$ *for* $1 \le i \le k-4$.

*Proof* In view of Lemma 2.11, it can be assumed that $k \ge 3$. Let $v_j(K)$ denote the $j$th Wu class of $K$. The orientability of $K$ implies that

$$w_1(K) = 0, \tag{1}$$

and hence by Wu's theorem [52, Theorem 11.14], $v_1(K) = 0$ and $v_2(K) = w_2(K)$. In particular,

$$\mathrm{Sq}^1(a) = v_1(K) \cup a = 0 \quad \text{for all} \quad a \text{ in } H^{k-1}(K; \mathbb{Z}/2), \tag{2}$$

$$\mathrm{Sq}^2(b) = v_2(K) \cup b = w_2(K) \cup b \quad \text{for all } b \text{ in } H^{k-2}(K; \mathbb{Z}/2). \tag{3}$$

Let $\kappa\colon K \hookrightarrow M$ be the inclusion map. Since $[K]_M$ is in $A_k$, for every cohomology class $z$ in $A^k$,

$$\langle \kappa^*(z), [K] \rangle = \langle z, \kappa_*([K]) \rangle = \langle z, [K]_M \rangle = 0. \tag{4}$$

According to (3), for every cohomology class $v$ in $H^{k-2}(M; \mathbb{Z}/2)$,

$$w_2(K) \cup \kappa^*(v) = \mathrm{Sq}^2(\kappa^*(v)) = \kappa^*(\mathrm{Sq}^2(v)).$$

If $\mathrm{Sq}^2(v)$ is in $A^k$ (which is automatically satisfied if $3 \le k \le 4$), then (4) gives

$$\langle w_2(K) \cup \kappa^*(v), [K] \rangle = 0. \tag{5}$$

According to (2), for every cohomology class $u$ in $H^{k-3}(M; \mathbb{Z}/2)$,

$$\mathrm{Sq}^1(w_2(K) \cup \kappa^*(u)) = 0.$$

On the other hand,

$$\mathrm{Sq}^1(w_2(K) \cup \kappa^*(u)) = \mathrm{Sq}^1(w_2(K)) \cup \kappa^*(u) + w_2(K) \cup \kappa^*(\mathrm{Sq}^1(u)).$$

Consequently,

$$\mathrm{Sq}^1(w_2(K)) \cup \kappa^*(u) = w_2(K) \cup \kappa^*(\mathrm{Sq}^1(u)).$$

By Wu's formula [52, p. 94], $\mathrm{Sq}^1(w_2(K)) = w_1(K) \cup w_2(K) + w_3(K)$, which in view of (1) gives $\mathrm{Sq}^1(w_2(K)) = w_3(K)$. Thus,

$$w_3(K) \cup \kappa^*(u) = \mathrm{Sq}^1(w_2(K)) \cup \kappa^*(u) = w_2(K) \cup \kappa^*(\mathrm{Sq}^1(u)).$$

If $\mathrm{Sq}^2(\mathrm{Sq}^1(u))$ is in $A^k$ (which is automatically satisfied if $3 \leq k \leq 4$), then (5) gives

$$\langle w_3(K) \cup \kappa^*(u), [K]\rangle = 0. \tag{6}$$

Since $K$ is a boundary, the Stiefel–Whitney numbers of $K$ are all 0. Consequently, in view of (1), (4), (5) and (6), the submanifold $K$ is adapted to $A$, provided that either ($c_1$) or ($c_2$) is satisfied. □

The assumption in Lemmas 2.11 and 2.12 that $K$ be a boundary is not a serious limitation for applications, as demonstrated below.

The following is a simple modification of a deep result of Thom [62, Théorème II.26].

**Lemma 2.13** (cf. [40, Lemma 4.7]) *Let $M$ be a compact connected smooth manifold and let $k$ be a positive integer satisfying $2k \leq \dim M$. Then, each homology class in $H_k(M; \mathbb{Z}/2)$ is of the form $[K]_M$, where $K$ is a $k$-dimensional connected smooth submanifold of $M$. Moreover, $K$ can be chosen in such a way that it is a boundary.*

Under some additional assumptions, $K$ can be assumed to be orientable.

**Lemma 2.14** *Let $M$ be a compact connected smooth manifold and let $k$ be a positive integer satisfying $2k + 1 \leq \dim M$. Assume that the homology group $H_{k-1}(M; \mathbb{Z})$ has no 2-torsion. Then, each homology class in $H_{k-1}(M; \mathbb{Z}/2)$ is of the form $[K]_M$, where $K$ is a $k$-dimensional connected orientable smooth submanifold of $M$. Moreover, $K$ can be chosen in such a way that it is a boundary.*

*Proof* By the universal coefficient theorem, the reduction modulo 2 homomorphism $\rho : H_k(M, \mathbb{Z}) \to H_k(M, \mathbb{Z}/2)$ is surjective. Hence, each homology class $\alpha$ in $H_k(M; \mathbb{Z}/2)$ is of the form $\alpha = \rho(\beta)$ for some homology class $\beta$ in $H_k(M; \mathbb{Z})$. According to [29, Corollary 15.3], one can find a $k$-dimensional oriented compact smooth manifold $N$, a smooth map $f : N \to M$ and an integer $r$ such that $f_*(\mu_N) = (2r + 1)\beta$, where $\mu_N$ is the fundamental class of $N$ in $H_k(N, \mathbb{Z})$. Since $2k + 1 \leq m := \dim M$, the map $f$ can be assumed to be a smooth embedding (cf. [30, Theorem 2.13]). Hence, $P := f(N)$ is an orientable smooth submanifold of $M$ with $[P]_M = \alpha$. By joining the connected components of $P$ with $k$-dimensional tubes in $M$, one obtains an orientable connected smooth submanifold $L$ of $M$ satisfying $[L]_M = \alpha$. Let $U$ be an open subset of $M \setminus L$, diffeomorphic to $\mathbb{R}^m$. Let $L'$ be a smooth submanifold of $U$, diffeomorphic to $L$. By joining $L$ and $L'$ with a $k$-dimensional tube in $M$, one gets an orientable connected smooth submanifold $K$ of $M$ satisfying $[K]_M = \alpha$. By construction, $K$ is a boundary. □

One more observation is required for the proofs of the main results.

**Lemma 2.15** *Let $M$ be a compact smooth manifold and let $A$ be an $r$-admissible subring of $H^*(M; \mathbb{Z}/2)$, where $r$ is a positive integer. Then, $\mathrm{Sq}^i(A^j) \subseteq A^{i+j}$ for all nonnegative integers $i$ and $j$ with $j \leq r - 1$.*

*Proof* It suffices to observe that for each full subring $B$ of $H^*(M; \mathbb{Z}/2)$, one has $\mathrm{Sq}^i(B^j) \subseteq B^{i+j}$ for all nonnegative integers $i$ and $j$. □

*Proof of Theorem 1.1* It is already known that (a) implies (b). If $k$ is an integer satisfying $1 \leq k \leq r$, then according to Lemma 2.13, each homology class in $A_k$ is of the form $[K]_M$, where $K$ is a $k$-dimensional connected smooth submanifold of $M$ that is a boundary. By Lemmas 2.11 and 2.15, $K$ is adapted to $A$. Hence, (b) implies (a) in view of Theorem 2.10. □

*Proof of Theorem 1.7* If $k$ is an integer satisfying $1 \leq k \leq r$, then according to Lemma 2.13, each homology class in $A_k$ is of the form $[K]_M$, where $K$ is a $k$-dimensional connected smooth submanifold of $M$ that is a boundary. Moreover, according to Lemma 2.14, $K$ can be assumed to be an orientable manifold if the homology group $H_{k-1}(M; \mathbb{Z})$ has no 2-torsion. By Lemmas 2.11 and 2.15, $K$ is adapted to $A$ if $k$ is in $\{1, \ldots, r-2\} \cup \{2\}$. By Lemmas 2.12 and 2.15, $K$ is adapted to $A$ if $r \geq 4$ and either $A^{r-3} = 0$ or the homology group $H_{r-2}(M; \mathbb{Z})$ has no 2-torsion. The proof is complete in view of Theorem 2.10. □



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